

Escaping Continuity Within Alpha Many Steps

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Making Mistakes in The Wadge Game

Setting: Instance-Solution Problems

Consider a theorem of the form

$$(\forall X \in \mathbb{N}^{\mathbb{N}})(R_1(X) \rightarrow (\exists Y \in \mathbb{N}^{\mathbb{N}})R_2(X, Y)).$$

We can take R_1 to be the domain of a partial multi-valued function $P : \subset \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ and R_2 to be the graph of P .

- An *instance-solution problem* is a partial multi-valued function $P : \subset \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ (working in $\mathbb{N}^{\mathbb{N}}$ lets us use computability).
- We call the domain $\text{dom}(P)$ the set of *instances* of P .
- For each P -instance $X \in \text{dom}(P)$,
 - $P(X) \subset \mathbb{N}^{\mathbb{N}}$ is the set of *P -solutions* to X .

We can play the Wadge game for any instance-solution problem P :

- Player 1 must play an instance.
- Player 2 tries to play a solution.

The Wadge Game and the Discontinuity Problem

- Game: Player 1 lists an element of $\mathbb{N}^{\mathbb{N}}$,
- Player 2 wins if they can list an element of $\mathbb{N}^{\mathbb{N}}$ that is different from Player 1's.
- Player 1 may wait as long as they want in between bits (or forever!).
- We use the '#' character to represent Player 1 waiting.

Definition

Let $\text{DIS} : (\mathbb{N} \cup \{\#\})^{\omega} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be defined by

$$\text{DIS}(X) = \begin{cases} \mathbb{N}^{\mathbb{N}} & \text{if } X \text{ has cofinitely many } \# \text{'s.} \\ \mathbb{N}^{\mathbb{N}} \setminus \{W\} & \text{if removing all } \# \text{'s from } X \text{ yields } W. \end{cases}$$

Let's play the Wadge game for DIS. [► Rules](#)

A Benchmark

Example ($\text{ACC}_{\mathbb{N}}$ (All-or-Co-Unique Choice))

- $\text{ACC}_{\mathbb{N}} : \subset (\mathbb{N} \cup \{\#\})^{\mathbb{N}} \rightrightarrows \mathbb{N}$.

$$\text{ACC}_{\mathbb{N}}(X) = \begin{cases} \mathbb{N} & \text{if } X = \#^{\mathbb{N}} \\ \mathbb{N} \setminus \{i\} & \text{if } X = \#^k i \#^{\mathbb{N}} \text{ for some } i \neq \#, W \in (\mathbb{N} \cup \{\#\})^{\mathbb{N}} \end{cases}$$

- Instances: An enumeration of the complement of either $\mathbb{N}$ or $\mathbb{N}$ without one element.
- Solutions: An element of the instance.

Let's play the Wadge game for $\text{ACC}_{\mathbb{N}}$.

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Let's play the Wadge game for $\text{ACC}_{\mathbb{N}}$.

Unlike DIS, we are irrevocably lost after one “mistake” in the Wadge game for $\text{ACC}_{\mathbb{N}}$.

More Possible Mistakes

Example (Having α -many attempts to solve $\text{ACC}_{\mathbb{N}}$)

Fix $\alpha < \omega_1$. $\text{ACC}_{\mathbb{N}}^{\alpha}$ is problem of having α -many chances to solve $\text{ACC}_{\mathbb{N}}$, with a new instance being given each time.

- Let's play the Wadge game for $\text{ACC}_{\mathbb{N}}^{\omega+1}$.
- We will let player 2 choose the next ordinal $\beta < \alpha$ when they are wrong.
- (This is okay because we require instances of $\text{ACC}_{\mathbb{N}}^{\alpha}$ to code the next ordinal into the number of $\#$'s so that Player 2 can wait long enough to ensure that the next ordinal is high enough.)

Outline

Observation

In the Wadge game:

DIS: P2 can always recover from any number of “mistakes”

$\text{ACC}_{\mathbb{N}}$: After one “mistake”, $P2$ is irrecoverably lost.

$\text{ACC}_{\mathbb{N}}^{\alpha}$: P2 can make up to α -many mistakes before no longer having solutions to work towards.

Idea

Capture this property. What does it mean?

The results obtained from pursuing this idea:

- An understanding of local continuity at a point
- A more suitable game for isolating this behavior
- Characterizations in terms of both continuous Weihrauch reducibility and Cantor-Bendixson rank.

Continuity at a point

Continuity of Problems

Let $P : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be a problem.

- We say that P is *continuous* (or continuously realizable) if there is a continuous partial function $\Theta : \subseteq \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ such that $\Theta(X) \in P(X)$ for all $X \in \text{dom}(P)$.
- We call such functions *continuous realizers* of P .

Theorem (Folklore)

P is continuous if and only if Player 2 wins the Wadge game for P .

Example

Each of the problems DIS and $\text{ACC}_{\mathbb{N}}^{\alpha}$ is discontinuous.

Local Discontinuities

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- I don't know of an ϵ - δ style characterization of realizer-based-continuity at a point for multivalued functions.
- Next best thing: use local continuity.

Definition

Let $P : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be a problem and fix $X \in \text{dom}(P)$. We say that P is *locally continuous* (or simply *continuous*) at X if there is open set $U \ni X$ such that $P|_U$ is continuous (i.e., $P|_U$ has a continuous realizer).

Local Discontinuities

Example

Fix $\alpha < \omega_1$. The point $\#^\omega$ is the only point of discontinuity of both $\text{ACC}_{\mathbb{N}}^\alpha$ and DIS.

Local realizers imply a global realizer.

Theorem

A problem $P : \subset \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ is continuous if and only if it is locally continuous at each point of its domain.

Proof.

($\implies$) P continuous implies that P is locally continuous at every point.

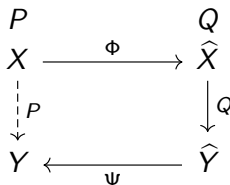
($\impliedby$) Suppose that P is continuous at every point of its domain.

Brouwer's bar theorem: An open cover of $A \subset \mathbb{N}^{\mathbb{N}}$ can be refined into a cover of disjoint clopen sets.

We apply the bar theorem on $\text{dom}(P)$ and then paste together the realizers on the disjoint clopen sets.



Weihrauch Reducibility



Let P and Q be instance-solution problems. We say that P is *continuously Weihrauch reducible* to Q ($P \leq_W^* Q$) if

- There are continuous functions $\Phi, \Psi : \mathbb{N}^{\mathbb{N}} \rightarrow \mathbb{N}^{\mathbb{N}}$ such that
- For each $X \in \text{dom}(P)$, we have that $\Phi(X) =: \hat{X} \in \text{dom}(Q)$ and
- For each $\hat{Y} \in Q(\Phi(X))$, we have that $\Psi(X, \hat{Y}) \in P(X)$.

Theorem

If $P \leq_W^* Q$ via forward function Φ and backward function Ψ and $X \in \text{dom}(P)$ is a discontinuity of P , then $\hat{X} := \Phi(X)$ is a discontinuity of Q .

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Proof.

Suppose that Q is continuous at $\hat{X}$ via Θ realizing $Q|_U$ on some open neighborhood $U \ni \hat{X}$. Then, $\Psi \circ (\text{id}, \Theta \circ \Phi)$ is a continuous realizer of $P|_{\Phi^{-1}(U)}$. □

Theorem (Nobrega and Pauly (2019))

For each P, Q , there is R such that $P \leq_W^* Q$ if and only if R is continuous. Furthermore, $P \leq_W Q$ if and only if Player 2 has a computable winning strategy in the Wadge game for R .

Ranks and Points of Discontinuity

Cantor-Bendixson Rank

For a point $X \in A \subset \mathbb{N}^{\mathbb{N}}$, we define $\text{rank}_A(X)$ recursively as follows.

- If X is isolated in A , then $\text{rank}_A(X) = 0$.
- If every initial segment $\sigma \sqsubset X$ extends to a non- X point in A of rank $\geq \alpha$ then we say that $\text{rank}_A(X) > \alpha$.
- Set $\text{rank}_A(X)$ to be the least ordinal α such that $\text{rank}_A(X) \not> \alpha$.

Goal

Describe points of discontinuity using Cantor-Bendixson rank.

Points of Sequential Discontinuity

Theorem (Pauly and Soldà, 2024)

Let $P : \subset \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be a problem. The following are equivalent.

- 1 There is a convergent sequence $a_1, a_2, \dots \rightarrow a \in \mathbb{N}^{\mathbb{N}}$ such that $P|_{\{a_1, a_2, \dots\} \cup \{a\}}$ is discontinuous
- 2 $P \geq_W^* \text{ACC}_{\mathbb{N}}$.
- 3 The first order part of P is discontinuous.

In terms of points of continuity:

- Isolated points of the domain are always points of continuity, so the limit point a is the only possible point of discontinuity for $P|_{\{a_1, a_2, \dots\} \cup \{a\}}$.
- a is the prototypical rank 1 discontinuity. It has Cantor-Bendixson rank 1 in $\{a_1, a_2, \dots\} \cup \{a\}$.

Pauly and Soldà conjecture that this is the $n = 1$ case of a larger set of results.

A Central Lemma

Lemma (Pauly and Soldà, 2024)

Let $P : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be a multivalued function. Suppose that there is a convergent sequence of points $a_1, a_2, \dots \rightarrow a \in \mathbb{N}^{\mathbb{N}}$ such that

$P|_{\{a\} \cup \{a_i : i \in \mathbb{N}\}}$ is not continuous.

Then, for each $Y \in P(a)$ there is $\rho \sqsubset Y$ such that for all $\sigma \sqsubset a$ there is $a_i \sqsupset \sigma$ such that $P(a_i) \cap \llbracket \rho \rrbracket = \emptyset$.

Playing the $\rho \sqsubset Y$ in the Wadge game is the “mistake”.

Lemma

Let $P : \subseteq \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be a problem and let $X \in \text{dom}(P)$. TFAE:

- 1 X is a discontinuity of P .
- 2 For each $Y \in P(X)$ there is $\rho \sqsubset Y$ such that for all $\sigma \sqsubset X$ there is τ with $\sigma \sqsubset \tau \not\sqsubset X$ such that $P|_{\llbracket \tau \rrbracket \times \llbracket \rho \rrbracket}$ is not continuous.

Note: Here we allow $P|_{\llbracket \tau \rrbracket \times \llbracket \rho \rrbracket}$ to have instances without solutions.

Ranking Discontinuities

Definition (Rank of discontinuity; sketch)

Let $P : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be a problem and let $X \in \text{dom}(P)$ be a discontinuity of P . We assign a countable ordinal rank of discontinuity to X as follows.

- If $P(X)$ is empty then we say that X is a rank-0 discontinuity of P .
- X is a rank α discontinuity of P if α is the least ordinal such that we can choose τ and ρ in the central lemma such that $P|[\llbracket \tau \rrbracket \times \llbracket \rho \rrbracket]$ has a discontinuity of lower rank.
- If there is no such ordinal α then we say that X is a rank- ∞ discontinuity of P .

Examples: $\#^{\omega}$ is a rank- α discontinuity of $\text{ACC}_{\mathbb{N}}^{\alpha}$ and is a rank- ∞ discontinuity of DIS.

Rank- α Discontinuity Game

Fix a problem $P : \subset \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ and countable ordinal α . The *rank- α discontinuity game for P* is a two player game in which Player I has α many steps to thwart Player II's proposed solutions to P .

- Turns are divided into stages a and b . Each Player takes a turn at each stage.
- On Turn 0.a, Player I first plays X_0 and $\gamma_0 = \alpha$. Then, Player II plays some $Y_0 \in P(X_0)$.
- On Turn 0.b, Player I plays some initial segment $\rho_0 \sqsubset Y_0$. Then, Player II plays some initial segment $\sigma_0 \sqsubset X_0$.
- Before Turn 1.a, Player 1 picks a $\gamma_1 < \alpha$. Then, play proceeds as in the rank- γ_1 discontinuity game on $P|[\sigma_0 \times \rho_0]$.

Winning conditions:

- Player I wins if Player II cannot make a move on their turn. This happens iff $P(X_n) \cap \llbracket \rho_{n-1} \rrbracket = \emptyset$.
- Player II wins if Player I cannot make a move on their turn. This happens iff $\gamma_n = 0$ on turn $(n+1).a$.

Rank- α Discontinuity Game

Fix a problem $P : \subset \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ and countable ordinal α . The *rank- α discontinuity game* for P is a two player game in which Player I has α many steps to thwart Player II's proposed solutions to P .

Turn		Player 1	Player 2
0	a	$\mathbf{X}_0 \in \text{dom}(P), \gamma_0 = \alpha$	$\mathbf{Y}_0 \in P(X_0)$
	b	$\rho_0 \sqsubset Y_0$	$\sigma_0 \sqsubset X_0$
1	a	$\sigma_0 \sqsubset \mathbf{X}_1 \in \text{dom}(P), \gamma_1 < \gamma_0$	$\mathbf{Y}_1 \in P(X_1) \cap \llbracket \rho_0 \rrbracket$
	b	$\rho_0 \sqsubseteq \rho_1 \sqsubset Y_1$	$\sigma_0 \sqsubseteq \sigma_1 \sqsubset X_1$
...	...	...	...
	...	...	...
$n+1$	a	$\sigma_n \sqsubset \mathbf{X}_{n+1} \in \text{dom}(P), \gamma_{n+1} < \gamma_n$	$\mathbf{Y}_{n+1} \in P(X_{n+1}) \cap \llbracket \rho_n \rrbracket$
	b	$\rho_n \sqsubseteq \rho_{n+1} \sqsubset Y_{n+1}$	$\sigma_n \sqsubseteq \sigma_{n+1} \sqsubset X_{n+1}$

Winning states:

- Player I wins if Player II cannot make a move on their turn. This happens iff $P(X_n) \cap \llbracket \rho_{n-1} \rrbracket = \emptyset$ at stage b of turn n .
- Player II wins if $\gamma_n = 0$ on turn $(n+1).a$.

Equivalence of These Notions

Theorem

Fix $P : \mathbb{C} \rightarrow \mathbb{N}^{\mathbb{N}} \Rightarrow \mathbb{N}^{\mathbb{N}}$. The following are equivalent.

- $P \geq_W^* \text{ACC}_{\mathbb{N}}^{\alpha}$.
 - Player 1 has a winning strategy in the rank- α discontinuity game for P .
 - P has a rank- β discontinuity for some $\beta \leq \alpha$.
 - P is discontinuous on a set of pointwise Cantor-Bendixson rank $\leq \alpha$.
-
- This satisfies part of Pauly and Soldà's conjecture. We still need to generalize the first order part. Current candidate: asking α -many first order questions.
 - The game is determined. This implies that Player 2 having a winning strategy implies that P is continuous on all sets of pointwise rank $\leq \alpha$.

We also have a pointwise version:

Theorem

Let $P : \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be a problem, let $X \in \text{dom}(P)$, and let α be a countable ordinal. The following are equivalent.

- ① *$P \geq_W^* \text{ACC}_{\mathbb{N}}^{\alpha}$ via forward function Φ such that $\Phi(\#^{\mathbb{N}}) = X$.*
- ② *There is a set $A \subset \mathbb{N}^{\mathbb{N}}$ such that $P|_A$ is discontinuous, X is a point of discontinuity of $P|_A$, and X has Cantor-Bendixson rank $\text{rank}_A(X) \leq \alpha$.*
- ③ *Player 1 has a winning strategy in the rank- α discontinuity game for P with initial move X .*
- ④ *X is a rank- β discontinuity of P for some $\beta \leq \alpha$.*

Minimal Discontinuous Problems

Minimality Results:

- Every problem is continuous on a set whose points all have CB-rank 0.
- $\text{ACC}_{\mathbb{N}}$ is the $\leq_W^*$ -least problem discontinuous on a set whose points all have CB rank ≤ 1 (Pauly and Soldà, 2024)
- $\text{ACC}_{\mathbb{N}}^\alpha$ is the $\leq_W^*$ -least problem discontinuous on a set whose points all have CB rank $\leq \alpha$.
- (ZF + DC + AD): DIS is the $\leq_W^*$ -least problem discontinuous on $\mathbb{N}^{\mathbb{N}}$ (Brattka, 2023).
- (ZF + DC + AD): For a represented space $\mathbb{X}$, the inequality problem $\neq_{\mathbb{X}}: \subset \overline{\mathbb{X}} \rightrightarrows \mathbb{X}$ on $\mathbb{X}$ is the $\leq_W^*$ least problem whose co-domain is $\mathbb{X}$ (Pauly and Soldà, 2024).

Proposition

Suppose that a is a rank- α discontinuity of P and suppose that $P \leq_W^* Q$ with forward function Φ . Then, $\Phi(a)$ is a rank- β discontinuity of Q for some $\beta \leq \alpha$.

Proof: Chase the table:

Turn		Player 1	Player 2
0	a	$\mathbf{X}_0 \in \text{dom}(P), \gamma_0 = \alpha$	$\mathbf{Y}_0 \in P(X_0)$
	b	$\rho_0 \sqsubset Y_0$	$\sigma_0 \sqsubset X_0$
1	a	$\sigma_0 \sqsubset \mathbf{X}_1 \in \text{dom}(P), \gamma_1 < \gamma_0$	$\mathbf{Y}_1 \in P(X_1) \cap \llbracket \rho_0 \rrbracket$
	b	$\rho_0 \sqsubseteq \rho_1 \sqsubset Y_1$	$\sigma_0 \sqsubseteq \sigma_1 \sqsubset X_1$
...	...	...	...
	...	...	...
$n+1$	a	$\sigma_n \sqsubset \mathbf{X}_{n+1} \in \text{dom}(P), \gamma_{n+1} < \gamma_n$	$\mathbf{Y}_{n+1} \in P(X_{n+1}) \cap \llbracket \rho_n \rrbracket$
	b	$\rho_n \sqsubseteq \rho_{n+1} \sqsubset Y_{n+1}$	$\sigma_n \sqsubseteq \sigma_{n+1} \sqsubset X_{n+1}$

Mistakes Target Changes In Wadge Game?

Definition (Player 1 Strategies Which Change Targets α -Many Times)

- Call a Player 1 strategy *rank-0* if it always plays the same point $X \in \mathbb{N}^{\mathbb{N}}$ no matter what Player 2 does.
- Call a Player 1 strategy *rank- α* if α is least such that there is a point $X \in \mathbb{N}^{\mathbb{N}}$ and a set of nodes $S \subset \mathbb{N}^{<\mathbb{N}}$ such that Player 1 plays initial segments of X until they see Player 2 play some $\rho \in S$, at which point they start using a rank- β strategy for some $\beta < \alpha$.

Proposition

Let $P : \subset \mathbb{N}^{\mathbb{N}} \rightrightarrows \mathbb{N}^{\mathbb{N}}$ be a problem. The following are equivalent.

- Player 1 has a winning rank- α strategy in the Wadge game for P .
- Player 1 has a winning strategy in the rank- α discontinuity game for P .

Thank You!

Wadge Game Rules

The *Wadge game* for P : a perfect-information game in which

- Player I and Player II alternately play natural numbers.
- Player I builds an input $p \in \mathbb{N}^{\mathbb{N}}$; Player II builds an output $q \in \mathbb{N}^{\mathbb{N}}$.
- Player II *wins* the run iff $p \notin \text{dom}(P)$ or $q \in P(p)$.
- Player I *wins* the run iff $p \in \text{dom}(P)$ and $q \notin P(p)$.

Theorem (Folklore; Nobrega, 2018)

P is continuous if and only if Player II has a winning strategy in the Wadge game for P .